

Twist-related geometries on q-Minkowski space

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Abstract

The role of the quantum universal enveloping algebras of symmetries in constructing non-commutative geometry of the space-time including vector bundles, measure, equations of motion and their solutions is discussed. In the framework of the twist theory the Klein-Gordon-Fock and Dirac equations on the quantum Minkowski space are studied from this point of view for the simplest quantum deformation of the Lorentz algebra induced by its Cartan subalgebra twist.

¹Partially supported by the RFFI grant 98-01-00310.

1 Introduction

The problem of building non-commutative models of the space-time arose in connection with attempts to formulate quantum field theory on this new basis, that could be free from difficulties inherent to the classical differential geometry description. Numerous publications on that matter address it in various aspects, using deformation of the group of relativistic symmetries as the starting point for further considerations. The first step on that way is to define the quantum group and its comodules along the line of FRT method [1]. At the next stage differential calculus consistent with the group coaction is constructed, following [2, 3, 4], and the equations of motion and observable algebra representations are studied [5, 6, 7, 8, 9, 10]. The programme just outlined thus gives the priority to the concept of group in proceeding to non-commutative geometry of the space time. In the present paper such a transition is performed on the basis of twist [11, 12] of a universal enveloping Lie algebra as the algebra of symmetries of a geometrical space. Twist preserves algebraic structure of a quantum algebra but changes its comultiplication. Twist is accompanied by deformation of the product in the algebra of functions on quantum group, and that process is entirely formulated in terms of the universal twisting 2-cocycle. This observation can be put into the foundation of a systematic approach to study whole classes of geometries corresponding to twist-related Hopf algebras. Quantum deformations of the Lorentz algebra are known to possess rich twist structure [13], and there are explicitly built the twisting cocycles for certain quantum-deformed Poincaré algebras [6, 15], so it is natural to apply this machinery to investigate properties of the deformed space-time.

Recall some principal facts concerning the subject under consideration. Let \mathcal{H} be a Hopf algebra possessing an invertible element (twist)

$$\Phi = \sum_i \Phi_i^{(1)} \otimes \Phi_i^{(2)} = \Phi_1 \otimes \Phi_2 \in \mathcal{H} \otimes \mathcal{H},$$

which satisfies the equation [12]

$$(\Delta \otimes id)(\Phi)\Phi_{12} = (id \otimes \Delta)(\Phi)\Phi_{23} \quad (1)$$

and the normalizing condition

$$(\varepsilon \otimes id)(\Phi) = 1 = (id \otimes \varepsilon)(\Phi). \quad (2)$$

It allows to define a Hopf algebra $\tilde{\mathcal{H}}$, which coincides with \mathcal{H} as an associative algebra but has the coproduct

$$\tilde{\Delta}(h) = \Phi^{-1}\Delta(h)\Phi. \quad (3)$$

The counit remains unchanged due to (2). The new antipode differs from the old one via the similarity transformation $\tilde{S}(h) \equiv uS(h)u^{-1}$ where $u = (\Phi_1^{-1})S(\Phi_2^{-1})$. Provided algebra \mathcal{H} is quasitriangular, so will be $\tilde{\mathcal{H}}$ and the new universal R-matrix will be expressed through the old one R and the twisting cocycle: $\tilde{R} = \Phi_{21}^{-1}R\Phi$. Twist establishes an equivalence relation among Hopf algebras which is not an isomorphism in the common sense as the twisted coproduct may differ significantly from original, e. g. be no more cocommutative. Twist-equivalence, having all the necessary features (symmetry, reflexivity, and transitivity), manifests itself in the equivalence of the monoidal categories of Hopf algebra representations, and also in relations between objects of the corresponding geometrical spaces, that is the subject of the present paper.

A specific case of the twist construction very much involved in the theory of quantum Lorentz algebra deformations is a twisted tensor product of two Hopf algebras [14]. Let \mathcal{H} be the ordinary tensor product $\mathcal{A} \otimes \mathcal{B}$, and an element $\Phi \in \mathcal{A} \otimes \mathcal{B} \sim \mathcal{A} \otimes 1 \otimes 1 \otimes \mathcal{B} \subset \mathcal{H} \otimes \mathcal{H}$ satisfies the factorizability conditions

$$\Delta_{\mathcal{A}}(\Phi) = \Phi_{13}\Phi_{23} \in \mathcal{A} \otimes \mathcal{A} \otimes \mathcal{B}, \quad \Delta_{\mathcal{B}}(\Phi) = \Phi_{13}\Phi_{12} \in \mathcal{A} \otimes \mathcal{B} \otimes \mathcal{B} \quad (4)$$

Then Φ is a solution to the equation (1), and in this case twist $\tilde{\mathcal{H}}$ of the original algebra with Φ will be denoted $\mathcal{A}^{\Phi} \otimes \mathcal{B}$.

Recall that the $*$ -operation (real form) on a Hopf algebra \mathcal{H} means antilinear involutive algebra anti-automorphism and coalgebra automorphism. Due to the identity $S* = *S^{-1}$, which always holds due to the uniqueness of the antipode, one can re-define the real form as $S^{2n}*$ for any integer number n . We can also consider homomorphic and anti-cohomomorphic antilinear operations of the kind $\theta = S^{2n+1}*$. Necessity of introducing them is accounted for the following. If some \mathcal{H} -comodule algebra \mathcal{A} with coaction $\beta: \mathcal{A} \rightarrow \mathcal{H} \otimes \mathcal{A}$ is endowed with anti-involution $a \rightarrow \bar{a}$ consistent with β , it must satisfy the equality $\beta(\bar{a}) = (* \otimes *)\beta(a)$. In this case \mathcal{H} plays the role of the function algebra on a quantum group of transformation of some manifold. The role of the universal enveloping algebra \mathcal{H}^* is different from the geometrical point of view because it does act on \mathcal{A} . To ensure consistency between the real forms and the action one has to require $\overline{(ha)} = S(h^*)\bar{a}$, $h \in \mathcal{H}^*$, $a \in \mathcal{A}$. So, further on we will mean by the real form of a quantum algebra a homomorphic and anti-cohomomorphic antilinear involution $\theta = S \circ *$.

It is important for the further geometrical considerations to investigate how real forms survive twist. From the formula (3) it follows that, condition

$$(\theta \otimes \theta)(\Phi) = \tau(\Phi), \quad (5)$$

fulfilled, the same involution is defined on $\tilde{\mathcal{H}}$. For the $*$ -operation the analogous natural requirement is

$$\Phi^* = \Phi^{-1}. \quad (6)$$

However, condition (5) is not always the case. Take for example the jordanian quantization of the Borel $sl(2)$ subalgebra. It is generated by two elements H and X^- subject to relation $[H, X^-] = -2X^-$, and the coproduct acquires a more simple

form after transition to the generator $\sigma = -\frac{1}{2}\ln(1 - 2X^-)$: $\tilde{\Delta}(\sigma) = \sigma \otimes 1 + 1 \otimes \sigma$, $\tilde{\Delta}(H) = H \otimes e^{2\sigma} + 1 \otimes H$. The deformation is induced by twist with the element $\Phi = e^{-H \otimes \sigma}$. It is easy to see that the classical $*$ -operation given by correspondence $H^* = -H$, $(X^-)^* = X^-$ can be brought through the twist because of (6). But for $\theta = S*$ one has $(\theta \otimes \theta)(\Phi) = \Phi^{-1} \neq \tau(\Phi)$, thus the same θ cannot be defined on the twisted algebra. Yet such a real form does exist and one can see that specifying it as $\sigma \rightarrow -\sigma$, $H \rightarrow He^{-2\sigma}$. The problem consists in establishing a correspondence between this involution and classical θ . It is clear that we must find an automorphism transforming the twisting element in such a way that it should contain the factor $\tau(\Phi)$. The problem is typical for the class of jordanian-like quantum algebras studied in [15, 16, 17]. The key role in resolving it belongs to the identity $(u \otimes u)\tau(S \otimes S)(\Phi) = \Phi^{-1}\Delta(u)$ fulfilled for any solution to the twist equation [12] (the element u is exactly the same like that taking part in the definition of the twisted antipode). Let element Φ and $*$ -operation obey (6). Then we have

$$\tau(\theta \otimes \theta)(\Phi) = \tau(S \otimes S)(\Phi^*) = \tau(S \otimes S)(\Phi^{-1}) = \Delta(u^{-1})(\Phi)(u \otimes u). \quad (7)$$

The direct calculation shows that the mapping $\tilde{\theta}$ given by the formula $\tilde{\theta}(h) = u\theta(h)u^{-1}$, is anti-cohomomorphic:

$$(\tilde{\theta} \otimes \tilde{\theta})(\tilde{\Delta}(h)) = \tau\tilde{\Delta}(\tilde{\theta}(h)).$$

Thus $\tilde{\theta}$ is an antilinear involutive algebraic and anti-coalgebraic automorphism of the algebra $\tilde{\mathcal{H}}$.

Quantizations of the Lorentz algebra $U(so(1, 3))$ are in close relations with quantum algebra $U_q(sl(2))$, so we remind here the basic facts of that theory.

The standard Drinfeld-Jimbo solution $U_q(sl(2))$ is build on the generators H and X^\pm such that

$$[H, X^\pm] = \pm 2X^\pm, \quad [X^+, X^-] = \frac{\text{sh}(\lambda H)}{\text{sh}(\lambda)},$$

$$\Delta(H) = 1 \otimes H + H \otimes 1, \quad \Delta(X^\pm) = e^{-\frac{\lambda}{2}H} \otimes X^\pm + X^\pm \otimes e^{\frac{\lambda}{2}H},$$

where we set $q = e^\lambda$. The universal R-matrix is [18]

$$\mathcal{R}_q = e^{\frac{\lambda}{2}H \otimes H} \sum_{n=0}^{\infty} \frac{(1 - q^{-2})^n}{[n]_q!} q^{\frac{n(n-1)}{2}} (e^{\frac{\lambda}{2}H} X^+ \otimes e^{-\frac{\lambda}{2}H} X^-)^n.$$

Algebra $U_q(sl(2))$ has three different real forms determined by their action on the generators as

$$\begin{aligned} \theta_{sl(2, \mathbf{R})} &: H \rightarrow H, \quad X^\pm \rightarrow X^\pm, \quad \lambda \in i\mathbf{R}, \\ \theta_{su(2)} &: H \rightarrow -H, \quad X^\pm \rightarrow -X^\mp, \quad \lambda \in \mathbf{R}, \\ \theta_{su(1,1)} &: H \rightarrow -H, \quad X^\pm \rightarrow X^\mp, \quad \lambda \in \mathbf{R}, \end{aligned}$$

and homomorphically extended over whole the algebra.

Another possible solution $U_h(sl(2))$ (we use definition $h = e^\xi$, which differs from generally accepted) is called the jordanian quantization and obtained by twist of the classical algebra [19] with the element $\Phi = e^{-\xi H \otimes \sigma}$. Expression of σ through X^- is given above where we discuss real forms of the Borel subalgebra of $sl(2)$. Here we introduce deformation parameter ξ that results in substitution $(X^-, \sigma) \rightarrow (\xi X^-, \xi \sigma)$ of the previously specified generators. The universal R-matrix for the jordanian quantization is

$$\mathcal{R}_h = \exp(\xi \sigma \otimes H) \exp(-\xi H \otimes \sigma). \quad (8)$$

The only real form $\tilde{\theta}_{sl(2, \mathbf{R})}$ for imaginary ξ is expressed through the classical involution $H \rightarrow H, X^\pm \rightarrow X^\pm$ by means of the element $u = \sum_{n=0}^{\infty} \frac{(-\xi)^n}{n!} H^n \sigma^n$. The similarity transformation with the element $e^{\alpha H}$ turns algebra $U_h(sl(2))$ into $U_{e^{2\alpha h}}(sl(2))$, so all the jordanian quantizations with non-zero deformation parameters are isomorphic to each other.

Classification of the quantum Lorentz group has been developed in [20, 21]. Below is the list of quantum Lorentz algebras dual to those solutions of [21] which have the classical limit.

1. Algebra $U_{q,r}(so(1, 3))$ is built as the twisted tensor product $U_q(sl(2)) \overset{\Phi}{\otimes} U_{\bar{q}^{-1}}(sl(2))$ $\Phi = r^{H_1 \otimes H_2}$. For real values of r the real form is defined by the equality $\theta = \tau(\theta_{sl(2, \mathbf{R})} \otimes \theta_{sl(2, \mathbf{R})})$, where τ denotes the operation of permutation between the left and right $sl(2)$ subalgebras. If $q = 1$, this solution is the twist of classical Lorentz algebra induced by the corresponding deformation of its Cartan subalgebra.
2. As algebras over the complex field, $U_q^\pm(so(1, 3))$ are isomorphic to the twisted tensor square $U_q(sl(2)) \overset{\Phi}{\otimes} U_q(sl(2))$ with the universal matrix \mathcal{R}_q as the twisting element. For real parameter q there exists two different real forms $\theta^+ = \tau(\theta_{su(2)} \otimes \theta_{su(2)})$ and $\theta^- = \tau(\theta_{su(1,1)} \otimes \theta_{su(1,1)})$. This algebra was introduced in [22, 23].
3. Algebra $U_{h,r}(so(1, 3))$ is $U_h(sl(2)) \overset{\Phi}{\otimes} U_{\bar{h}^{-1}}(sl(2))$ with $\Phi = r^{\sigma_1 \otimes \sigma_2}$, where dependence on the parameter h can be eliminated by a change of the basis. As this solutions is a composition of two twists (the first one is the jordanian deformation of the left and right $sl(2)$ components), algebra $U_{h,r}(so(1, 3))$ is a pure twist of the classical Lorentz algebra. Its real form is given by $\theta = \tau(\tilde{\theta}_{sl(2, \mathbf{R})} \otimes \tilde{\theta}_{sl(2, \mathbf{R})})$, for real values of r .
4. Algebra $U_h(so(1, 3))$ can be represented as the tensor square of the jordanian $U_h(sl(2))$ twisted by its R-matrix taken as Φ . However, the Lorentz real form is easier described if we consider the direct twist of $U(so(1, 3))$ with the element $\Phi = \exp\{(H_1 + H_2) \otimes \frac{1}{2} \ln(1 - \xi(X_1^- + X_2^-))\}$. It is induced by the deformation of the subalgebra isomorphic to the $sl(2)$ Borel subalgebra, and the real form differs from classical only by the similarity transformation with the element u as discussed above. Parameter ξ is assumed to be imaginary.

Thus all the quantum Lorentz algebras belong to the two classes of twist-equivalence related to the standard and jordanian deformations. Twist takes part in quantization

of the Poincaré algebra as well [6, 15], although in this case the explicit description of twist-equivalence classes is as yet unknown.

2 Twisted modules

Twist preserves the multiplicative structure of Hopf algebras as well as their central elements. Tensor products of representations of twist-equivalent algebras are isomorphic too, and their spectra coincide. That does not make any difference between such Hopf algebras from the point of view of internal symmetries. Kinematical and dynamical consequences may be rather significant, however. Quasiclassical limit displays profoundly unusual behavior of particle trajectories on Minkowski space even for the simplest deformations [7]. So, throughout this paper one can imagine twist as a transformation of the classical algebra $U(so(1, 3))$.

Recall that a module-algebra \mathcal{A} for a Hopf algebra is endowed with a multiplication respected by the action of \mathcal{H} in the following sense [24]:

$$h(a \cdot b) = h^{(1)}a \cdot h^{(2)}b, \quad h \in \mathcal{H}, \quad a, b \in \mathcal{A}.$$

Here $h^{(1)} \otimes h^{(2)}$ denotes symbolically the coproduct $\Delta(h)$. This definition matches the algebra of functions on some manifold, or the differential operator algebra, or just the matrix ring of linear transformation on a vector space, where \mathcal{H} is the enveloping algebra of a classical group of automorphisms. Twist of \mathcal{H} provides a new algebraic structure on \mathcal{A} consistent with the action of $\tilde{\mathcal{H}}$. This new algebra coinciding with \mathcal{A} as an \mathcal{H} -module and denoted further on by $\tilde{\mathcal{A}}$, is equipped with the multiplication

$$a * b = \Phi_1 a \cdot \Phi_2 b, \quad a, b \in \mathcal{A}.$$

Associativity is guaranteed by equation (1). Note that in the case of function algebra on a classical Poisson manifold this construction provides an example of deformation

quantization introduced in [25].

Properties of two twist-equivalent q -manifolds may differ significantly. Nevertheless, there are transition rules between geometrical objects if they are somehow related to the symmetry algebra action. Hardly one may look forward to relate a representation of $\tilde{\mathcal{A}}$ to an arbitrary representation of \mathcal{A} as even their spectra may differ, e.g. due to commutativity violation. However, this is possible for \mathcal{H} -covariant representations meaning that both \mathcal{A} and \mathcal{H} act on the same linear space \mathcal{W} , and $\pi(ha) = \rho(h^{(1)})\pi(a)\rho(S(h^{(2)}))$ (notations π and ρ for \mathcal{A} - and \mathcal{H} -actions will be omitted further on). Within the classical differential geometry such representations correspond to vector bundles with a group leaf-wise action. In particular, the regular representation of \mathcal{A} on itself (trivial 1-bundle) by multipliers is \mathcal{H} -covariant. We may also think of \mathcal{A} as the differential operator algebra on the Minkowski space and of \mathcal{W} as the algebra of functions. Let us show that \mathcal{H} -covariant module \mathcal{W} over \mathcal{A} turns into $\tilde{\mathcal{H}}$ -covariant module $\tilde{\mathcal{W}}$ over $\tilde{\mathcal{A}}$, if action of $\tilde{\mathcal{A}}$ on $\tilde{\mathcal{W}}$ is defined as

$$a * w = \Phi_1 a \cdot \Phi_2 w, \quad a \in \mathcal{A}, \quad w \in \mathcal{W}.$$

To check this up, it is sufficient to notice that \mathcal{H} -covariance follows (and *vice versa*) from $h(a \cdot w) = h^{(1)}a \cdot h^{(2)}w$. The rest implications repeat those for the case $\mathcal{W} = \mathcal{A}$.

From geometry of vector bundles, let us proceed to a more detailed study of the geometry of the base space itself. If an algebra \mathcal{A} is endowed with an involutive antilinear anti-automorphism $a \rightarrow \bar{a}$, it can be transferred to $\tilde{\mathcal{A}}$, provided the twisting element satisfies (5). If that is not the case but identity (6) holds, the real form steel can be introduced on $\tilde{\mathcal{A}}$. First let us prove the following proposition

Lemma 1 *If the element Φ satisfies (6), the equality*

$$\theta(u) = u^{-1} \tag{9}$$

is true.

Using the permutation rule $*S = S^{-1}*$, we find

$$\theta(u) = \theta(\Phi_1^{-1})\theta S(\Phi_2^{-1}) = S((\Phi_1^{-1})_1^*)S[S(\Phi_2^{-1})]^* = S((\Phi_1^{-1})_1^*)(\Phi_2^{-1})_2^* = S(\Phi_1)\Phi_2 = u^{-1}.$$

The latter holds for any solution of the twist equation [12]. It follows immediately that (9) is true with respect to $\tilde{\theta}$, too. We define the twisted real form on the module-algebra $\tilde{\mathcal{A}}$ as $u\bar{f}$. It is easy to check that the operation thus introduced is involutive, anti-homomorphic and consistent with the action of $\tilde{\mathcal{H}}$.

Suppose now that \mathcal{A} possesses a measure μ , i.e. a linear functional positive on elements of the form $a \cdot \bar{a}$ (as the function algebra on a locally compact topological space does). The same measure is valid for $\tilde{\mathcal{A}}$. Indeed, we find $a * \bar{a} = \Phi_1 a \cdot \Phi_2 \bar{a} = \Phi_1 a \cdot \overline{(\theta(\Phi_2) a)}$. If identity (5) is fulfilled, the relation $\Phi_1 \otimes \theta(\Phi_2) = \theta(\Phi_2) \otimes \Phi_1$ holds as well, and, consequently, $\Phi_1 \otimes \theta(\Phi_2)$ can be represented by a sum $\sum \varphi_i \otimes \varphi_i$. Further, we have $a * \bar{a} = \sum \varphi_i a \cdot \overline{\varphi_i a}$, and therefore $\mu(a * \bar{a}) \geq 0$. In the case when (6) is true, one can extend the Hopf algebra adding the square root of u . It is straightforward that the composition of the coboundary twist with the element $\Delta(u^{-\frac{1}{2}})(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$ and successive twist with the element $(u^{-\frac{1}{2}} \otimes u^{-\frac{1}{2}})\Phi(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$ obeys (5). This double transformation is carried out by means of the 2-cocycle $\Delta(u^{-\frac{1}{2}})\Phi(u^{\frac{1}{2}} \otimes u^{\frac{1}{2}})$, and the required property (5) readily follows from (7). So, we can apply all the previous considerations to this composite twist, which differs from initial one by the internal automorphism of the Hopf algebra only.

By means of μ an integration and the Hermitean scalar product are introduced on $\tilde{\mathcal{A}}$, turning it into a pre-Hilbert space. The regular representation of the involutive algebra $\tilde{\mathcal{A}}$ comes out to be a homomorphism into the $*$ -operator algebra on that space. It is seen, that although integrals are not changed in proceeding to $\tilde{\mathcal{A}}$, the Hermitian form $(a, b) = \mu(a \cdot \bar{b})$ is undergone deformation. Isometries among module-algebras would remain so among their twisted counterparts, provided they are permutable with the action of $\tilde{\mathcal{H}}$. In particular, this is true for the classical Fourier transformation,

which is a homomorphism from the algebra of functions on the coordinate Minkowski space to the convolution algebra on the space of momenta.

3 Klein-Gordon-Fock equation

Taking into account considerations of the previous section, one can state close connection between structures on \mathcal{A} and $\tilde{\mathcal{A}}$. This conclusion is not of theoretical interest only, but yields a machinery to solve equations of motion, study physical states and so on. The principal observation is that, providing for some $c \in \mathcal{A}$ and all $h \in \mathcal{H}$ the condition $hc = \varepsilon(h)c$ is obeyed, the result of the action of c on the elements of the modules \mathcal{W} and $\tilde{\mathcal{W}}$, coinciding as linear spaces, will be the same: $c * w = (\Phi_1 c) \cdot (\Phi_2 w) = (\varepsilon(\Phi_1) c) \cdot (\Phi_2 w) = c \cdot w$ for arbitrary $w \in \mathcal{W}$. Analogously, if for some $w \in \mathcal{W}$ and any $h \in \mathcal{H}$ equality $hw = \varepsilon(h)w$ takes place, one has $c * w = c \cdot w$. In other words, the result of parings with an invariant element coincides in the twisted and untwisted cases. We immediately find that \mathcal{H} -invariant elements of the center of \mathcal{A} remain so for $\tilde{\mathcal{A}}$. If \mathcal{A} means the classical Minkowski space \mathcal{M} , any Lorentz invariant element will commute with the whole $\tilde{\mathcal{M}}$. So, for example, will do the invariant length.

Ordered homogeneous monomials $(z^{\mu_1} \cdot \dots \cdot z^{\mu_k})_{k=0,\dots,\infty}$ in coordinate functions z^μ on the classical Minkowsky space form a basis of the function algebra \mathcal{M} . Since the deformed product $*$ is expressed through \cdot by the invertible element Φ , the quadratic relations $z^\mu z^\nu = z^\nu z^\mu$ in \mathcal{M} go over into the quadratic relations $(\Phi_1^{-1} z^\mu) * (\Phi_2^{-1} z^\nu) = (\Phi_1^{-1} z^\nu) * (\Phi_2^{-1} z^\mu)$, due to the linear action of \mathcal{H} on coordinate functions.

Homogeneous monomials of degree n with respect to the twisted product are expressed through monomials of the same degree in terms of the old product:

$$z^{\mu_1} * \dots * z^{\mu_k} = \cdot(\Omega^k(z^{\mu_1} \otimes \dots \otimes z^{\mu_k})),$$

where elements $\Omega^k \in \mathcal{H}^{\otimes k}$ form the family of operators intertwining braid group B_n

representations related to the R-matrices \mathcal{R} and $\tilde{\mathcal{R}}$ [26]. Consequently, ordered monomials $(z^{\mu_1} * \dots * z^{\mu_k})_{k=0,\dots,\infty}$ form a basis in the twisted module $\tilde{\mathcal{M}}$ too, and functions on the quantum Minkowski space can be formally expanded over it. Thus, physically interesting Lorentz-invariant objects remain the same as elements of twisted modules however they have entirely different expression through the generators in terms of the twisted multiplication. Let us turn, for example, to the Klein-Gordon-Fock equation on the classical Minkowski space

$$(g^{\mu\nu} \partial_\mu \cdot \partial_\nu + m^2) f(z) = 0. \quad (10)$$

Since the D'Alembert operator is Lorentz-invariant, function $f(z)$, as belonging to $\tilde{\mathcal{M}}$, satisfies the same equation with D'Alembertian

$$\tilde{g}^{\mu\nu} \partial_\mu * \partial_\nu = g^{\mu\nu} (\Phi^{-1})_{\mu\nu}^{\rho\xi} \partial_\rho * \partial_\xi = g^{\mu\nu} \partial_\mu \cdot \partial_\nu.$$

Here $(\Phi)_{\mu\nu}^{\rho\xi}$ denotes the matrix of the action of Φ on partial derivatives ∂_μ . Thus twisting gives rise to redefinition of the metric. Now consider the function $e^{i(p,z)}$ on the cotangent bundle over the classical Minkowski space. The condition $g^{\mu\nu} p_\mu \cdot p_\nu = m^2$ imposed, it becomes an element of the algebra of functions on the mass shell, which is the quotient of $\tilde{\mathcal{M}}$ over the ideal $(g^{\mu\nu} p_\mu \cdot p_\nu - m^2)$. Then $e^{i(p,z)}$ is a solution to the quantum Klein-Gordon-Fock equation and represents a plane wave. Because of the Lorentz-invariance of (p, z) , power series in this element coincide both in the classical and twisted cases, while the canonical paring between linear coordinate space and that of momenta is deformed: $\tilde{g}_\nu^\mu p_\mu * z^\nu = \tilde{g}_\nu^\mu (\Phi_1 p_\mu) \cdot (\Phi_2 z^\mu) = g_\nu^\mu p_\mu \cdot z^\nu$. The general solution to (10) for the real scalar field can be decomposed into the sum of the plane waves

$$f(z) = \int dp \delta(p^2 - m^2) [a(p) e^{-i(p,z)} + a^\dagger(p) e^{i(p,z)}],$$

that now can be treated in terms of $\tilde{\mathcal{M}}$, with $\int dp \delta(p^2 - m^2)$ being the invariant measure on the quantum mass shell and the integrand rewritten using the new product $*$.

Let us show how this conclusion agrees with earlier results by [7] taking the simplest case of the algebra $U_{1;r}(so(1,3))$ as an example. We are interested in the three sets

$$Z = \begin{pmatrix} z^1 & z^4 \\ z^2 & z^3 \end{pmatrix}, \quad P = \begin{pmatrix} p_1 & p_2 \\ p_4 & p_3 \end{pmatrix}, \quad D = \begin{pmatrix} \partial_1 & \partial_2 \\ \partial_4 & \partial_3 \end{pmatrix},$$

representing, respectively, coordinates, momenta, and partial derivatives on \mathcal{M} , which are transformed under the Lorentz group coaction according to the rule

$$Z \rightarrow MZM^\dagger, \quad P \rightarrow (M^\dagger)^{-1}PM^{-1}, \quad D \rightarrow (M^\dagger)^{-1}DM^{-1}, \quad M \in SL(2).$$

Three classical Lorentz invariants are given by the bilinear forms

$$(z, z) = z^1 z^3 - z^2 z^4, \quad (p, p) = p_1 p_3 - p_2 p_4, \quad (p, z) = p_1 z^1 + p_2 z^2 + p_3 z^3 + p_4 z^4,$$

and the matrix commutation relations between D and Z read $D_1 Z_2 - Z_2 D_1 = \mathcal{P}$. Here \mathcal{P} is the permutation operator in $\mathbf{C}^2 \otimes \mathbf{C}^2$:

$$\mathcal{P} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Let us perform the twist of the classical universal enveloping Lorentz algebra with the element $r^{-H_2 \otimes H_1}$, where H_1 and H_2 are its Cartan generators and consider the resulting quantum Minkowski space. Action of H_1 and H_2 on Z and P is defined as

$$\begin{aligned} H_1: \begin{pmatrix} z^1 & z^4 \\ z^2 & z^3 \end{pmatrix} &\rightarrow \begin{pmatrix} z^1 & z^4 \\ -z^2 & -z^3 \end{pmatrix}, & H_1: \begin{pmatrix} p_1 & p_2 \\ p_4 & p_3 \end{pmatrix} &\rightarrow \begin{pmatrix} -p_1 & p_2 \\ -p_4 & p_3 \end{pmatrix}, \\ H_2: \begin{pmatrix} z^1 & z^4 \\ z^2 & z^3 \end{pmatrix} &\rightarrow \begin{pmatrix} z^1 & -z^4 \\ z^2 & -z^3 \end{pmatrix}, & H_2: \begin{pmatrix} p_1 & p_2 \\ p_4 & p_3 \end{pmatrix} &\rightarrow \begin{pmatrix} -p_1 & -p_2 \\ p_4 & p_3 \end{pmatrix}. \end{aligned}$$

From now on we adopt a convention of distinguishing the elements of \mathcal{M} , treated as elements of $\tilde{\mathcal{M}}$ by the tilde. Evaluating multiplication on $\tilde{\mathcal{M}}$ we find

$$\tilde{z}^\mu * \tilde{z}^\nu = a(\mu, \nu) z^\mu \cdot z^\nu, \quad \tilde{p}_\mu * \tilde{p}_\nu = a(\mu, \nu) p_\mu \cdot p_\nu,$$

$$\tilde{z}^\mu * \tilde{p}_\nu = b(\mu, \nu) z^\mu \cdot p_\nu, \quad \tilde{p}_\mu * \tilde{z}^\nu = b(\mu, \nu) p_\mu \cdot z^\nu,$$

where numbers $a(\mu, \nu)$ and $b(\mu, \nu) = (a(\mu, \nu))^{-1}$ form the matrices (the first index labels the rows)

$$a(\mu, \nu) = \begin{pmatrix} r^{-1} & r & r & r^{-1} \\ r^{-1} & r & r & r^{-1} \\ r & r^{-1} & r^{-1} & r \\ r & r^{-1} & r^{-1} & r \end{pmatrix}, \quad b(\mu, \nu) = \begin{pmatrix} r & r^{-1} & r^{-1} & r \\ r & r^{-1} & r^{-1} & r \\ r^{-1} & r & r & r^{-1} \\ r^{-1} & r & r & r^{-1} \end{pmatrix}.$$

This leads to the following relations among \tilde{z} and \tilde{p} :

$$\tilde{z}^\mu * \tilde{z}^\nu = a(\mu, \nu) b(\nu, \mu) \tilde{z}^\nu * \tilde{z}^\mu, \quad \tilde{p}_\mu * \tilde{p}_\nu = a(\mu, \nu) b(\nu, \mu) \tilde{p}_\nu * \tilde{p}_\mu,$$

$$\tilde{p}_\mu * \tilde{z}^\nu = b(\mu, \nu) a(\nu, \mu) \tilde{z}^\nu * \tilde{p}_\mu,$$

Explicitly, in the coordinate sector this reads [5]

$$\begin{aligned} \tilde{z}^1 * \tilde{z}^2 &= r^2 \tilde{z}^2 * \tilde{z}^1, & \tilde{z}^1 * \tilde{z}^3 &= \tilde{z}^3 * \tilde{z}^1, & \tilde{z}^4 * \tilde{z}^1 &= r^2 \tilde{z}^1 * \tilde{z}^4, \\ \tilde{z}^2 * \tilde{z}^3 &= r^2 \tilde{z}^3 * \tilde{z}^2, & \tilde{z}^2 * \tilde{z}^4 &= \tilde{z}^4 * \tilde{z}^2, & \tilde{z}^3 * \tilde{z}^4 &= r^2 \tilde{z}^4 * \tilde{z}^3. \end{aligned}$$

Twisting of the Lorentz algebra of the type being considered is induced by that of its Cartan subalgebra. The latter preserves two bilinear forms $z_1 \cdot z_3$ and $z_2 \cdot z_4$, and that explains why elements $\tilde{z}_1 * \tilde{z}_3$ and $\tilde{z}_2 * \tilde{z}_4$ belong to the center of $\tilde{\mathcal{M}}$.

Commutation relations involving partial derivatives $\tilde{\partial}_\mu$ are obtained in the similar way:

$$\tilde{\partial}_\mu * \tilde{z}^\nu = b(\mu, \nu) a(\nu, \mu) \tilde{z}^\nu * \tilde{\partial}_\mu + b(\mu, \nu), \quad \tilde{\partial}_\mu * \tilde{p}_\nu = a(\mu, \nu) b(\nu, \mu) \tilde{p}_\nu * \tilde{\partial}_\mu$$

$$\tilde{\partial}_\mu * \tilde{\partial}_\nu = a(\mu, \nu) b(\nu, \mu) \tilde{\partial}_\nu * \tilde{\partial}_\mu$$

Introducing the operator $V = r^{\sigma^3 \otimes \sigma^3}$ with the Pauli matrix $\sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, one can rewrite these relations in the compact form employed in [7]:

$$\tilde{Z}_1 V \tilde{Z}_2 = \tilde{Z}_2 V \tilde{Z}_1, \quad \tilde{D}_1 V^{-1} \tilde{D}_2 = \tilde{D}_2 V^{-1} \tilde{D}_1, \quad \tilde{P}_1 V^{-1} \tilde{P}_2 = \tilde{P}_2 V^{-1} \tilde{P}_1,$$

$$\tilde{P}_1 \tilde{Z}_2 = V \tilde{Z}_2 \tilde{P}_1 V^{-1}, \quad \tilde{D}_1 V^{-1} \tilde{P}_2 = \tilde{P}_2 \tilde{D}_1.$$

The Lorentz invariants turn into

$$\begin{aligned} (\tilde{z}, \tilde{z})_r &= (z, z) = r^{-1} \tilde{z}^1 * \tilde{z}^3 - r \tilde{z}^2 * \tilde{z}^4, \\ (\tilde{p}, \tilde{p})_r &= (p, p) = r^{-1} \tilde{p}_1 * \tilde{p}_3 - r \tilde{p}_2 * \tilde{p}_4, \\ (\tilde{p}, \tilde{z})_r &= (p, z) = r^{-1} \tilde{p}_1 * \tilde{z}^1 + r \tilde{p}_2 * \tilde{z}^2 + r^{-1} \tilde{p}_3 * \tilde{z}^3 + r \tilde{p}_4 * \tilde{z}^4. \end{aligned}$$

The commutation relations obtained agree with those deduced in [7], where the role of $\tilde{\partial}_\mu$ is played by quantum partial derivatives δ_μ . However, the matrix $\tilde{D}_1 \tilde{Z}_2 - V \tilde{Z}_2 \tilde{D}_1 V^{-1}$ turns out to be dependent on the deformation parameter

$$\begin{pmatrix} r & 0 & 0 & 0 \\ 0 & 0 & r^{-1} & 0 \\ 0 & r & 0 & 0 \\ 0 & 0 & 0 & r^{-1} \end{pmatrix},$$

in contrast to [7]. The correspondence $\delta_\mu \leftrightarrow \tilde{\partial}_\mu$ and interpretation of the relations involving δ_μ in terms of ∂ and $\tilde{\partial}$ thus require special investigation. Entities δ_i are introduced in [7] via the transformation law under the coaction $D \rightarrow (M^\dagger)^{-1} D M^{-1}$ of the Lorentz group. The commutation relations $M_1^\dagger V M_2 = M_2 V M_1^\dagger$ in the group determine the invariant uniquely modulo a factor. In terms of δ_μ that invariant is written in the form $\delta^2 = r \delta_1 \delta_3 - r^{-1} \delta_2 \delta_4$, whereas the twist of the algebra $U(so(1, 3))$

gives $r^{-1}\tilde{z}_1 * \tilde{z}_3 - r\tilde{z}_2 * \tilde{z}_4$. It is seen that $\delta^2 \quad (\tilde{p}, \tilde{p})_r$ have different dependence on r . On the other hand, deformation of the Lorentz metric in proceeding from \mathcal{M} to $\tilde{\mathcal{M}}$ implies that the basis $\tilde{\partial}_\mu$ is no more conjugate to the basis \tilde{z}^μ and matrices $\tilde{\partial}_\mu * \tilde{z}^\nu$ $\tilde{z}^\nu * \tilde{\partial}_\mu$ behave unsatisfactory under the Lorentz transformations. For the transition to the conjugate basis, one should set

$$(\delta_1, \delta_2, \delta_3, \delta_4) = (r^{-1}\tilde{\partial}_1, r\tilde{\partial}_2, r^{-1}\tilde{\partial}_3, r\tilde{\partial}_4).$$

Thus one comes to the right expression of the invariant $r\delta_1\delta_3 - r^{-1}\delta_2\delta_4 = r^{-1}\tilde{\partial}_1 * \tilde{\partial}_3 - r\tilde{\partial}_2 * \tilde{\partial}_4$. The matrix of relations between δ_μ and \tilde{z}^ν takes the required form independent² on r .

Deviation from commutativity in $\tilde{\mathcal{M}}$ may be accounted for the so called "undressing" transformation proposed in [7]. It employs two elements u and v , fulfilling the Weyl relations $vu = r^2uv$, $\bar{u} = u^{-1}$, $\bar{v} = v$. Coordinates and momenta are expressed through u and v and classical commutative generators

$$X = \begin{pmatrix} x^1 & x^4 \\ x^2 & x^3 \end{pmatrix}, \quad Y = \begin{pmatrix} y_1 & y_2 \\ y_4 & y_3 \end{pmatrix}$$

by the formulas

$$\tilde{Z} = \begin{pmatrix} vx^1 & u^{-1}x^4 \\ ux^2 & v^{-1}x^3 \end{pmatrix}, \quad \tilde{P} = \begin{pmatrix} v^{-1}y_1 & u^{-1}y_4 \\ uy_4 & vy_3 \end{pmatrix}.$$

Noncommutative elements u and v drop from the scalar products for which we obtain the following:

$$(z, z) = (\tilde{z}, \tilde{z})_r = (x, x)_r, \quad (p, p) = (\tilde{p}, \tilde{p})_r = (y, y)_r, \quad (p, z) = (\tilde{p}, \tilde{p})_r = (y, x)_r.$$

²Let us note that there is a freedom in the choice of the twisting element. Algebra $U_{1;r}(so(1,3))$ may be equally obtained if one takes $\Phi = e^{\alpha H_1 \otimes H_2 + \beta H_2 \otimes H_1}$, $\alpha - \beta = \ln(r)$. The form of the Lorentz invariant depends a particular choice of parameters α and β . For example, assuming $\alpha = -\beta$, they stay undeformed at all, and two bases (\tilde{z}^μ) and (\tilde{p}_μ) remain orthonormal and mutually conjugate.

These noncommutative elements also disappear from the D'Alembert operator, hence the function $\int d\tilde{\mu}(y)[\tilde{a}(y)e^{-i(y,x)_r} + h.c.]$ is the general solution to the quantum Klein-Gordon-Fock equation with the deformed metric. Proceeding to the orthonormal basis shows that this function is equal to $\tilde{f}(x)$, $f(z) = \int d\mu(p)[a(p)e^{-i(p,z)} + h.c.]$. Returning to non-commutative variables in $\tilde{f}(x)$, we find again that $\tilde{f}(\tilde{z}) = f(z)$ is the general solution to the quantum Klein-Gordon-Fock equation.

4 Dirac equation

In this paragraph we study four-component spinor fields which form a \mathcal{M} -module $\mathcal{W} = \mathcal{M} \otimes \mathbf{W}$ of sections of the trivial spinor bundle over the Minkowski space. The four-dimensional complex lineal \mathbf{W} is the space of the $(\frac{1}{2}, \frac{1}{2})$ -representation of the classical Lorentz algebra, and the module \mathcal{W} is freely generated by the basic elements $e^{(\alpha)} \in \mathbf{W}$:

$$e^{(1)} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad e^{(2)} = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \end{pmatrix}, \quad e^{(3)} = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 0 \end{pmatrix}, \quad e^{(4)} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 1 \end{pmatrix}.$$

The Cartan generators H_1 and H_2 are represented by the matrix

$$H_1 = \begin{pmatrix} -\sigma^3 & 0 \\ 0 & 0 \end{pmatrix}, \quad H_2 = \begin{pmatrix} 0 & 0 \\ 0 & \sigma^3 \end{pmatrix},$$

in the chiral basis diagonalizing the matrix γ^5 .

The space \mathcal{W} is also a module over the differential operator algebra $\text{Diff}(\mathcal{W})$, which is the tensor product of $\text{Diff}(\mathcal{M})$ by the ring $\text{Lin}(\mathbf{W}, \mathbf{W})$. In the twisted algebra $\widetilde{\text{Diff}}(\mathcal{W})$ linear operators on \mathbf{W} and partial derivatives do not commute with each other. Let us

calculate the commutation relations between the Dirac matrices and the generators \tilde{z} , \tilde{p} , and $\tilde{\partial}$:

$$\begin{aligned}\tilde{z}^\mu * \tilde{\gamma}^\nu &= a(\mu, \nu) b(\nu, \mu) \tilde{\gamma}^\nu * \tilde{z}^\mu, \quad \tilde{p}_\mu * \tilde{\gamma}^\nu = b(\mu, \nu) a(\nu, \mu) \tilde{\gamma}^\nu * \tilde{p}_\mu, \\ \tilde{\partial}_\mu * \tilde{\gamma}^\nu &= b(\mu, \nu) a(\nu, \mu) \tilde{\gamma}^\nu * \tilde{\partial}_\mu.\end{aligned}$$

The Dirac matrices themselves obey the relation

$$b(\mu, \nu) \tilde{\gamma}^\mu * \tilde{\gamma}^\nu + b(\nu, \mu) \tilde{\gamma}^\nu * \tilde{\gamma}^\mu = 2g^{\mu\nu},$$

that gives, in the complex basis of the Minkowski space being used,

$$\begin{aligned}\tilde{\gamma}^1 * \tilde{\gamma}^3 + \tilde{\gamma}^3 * \tilde{\gamma}^1 &= r, \quad r^{-1} \tilde{\gamma}^1 * \tilde{\gamma}^2 = -r \tilde{\gamma}^2 * \tilde{\gamma}^1, \quad r \tilde{\gamma}^1 * \tilde{\gamma}^4 = -r^{-1} \tilde{\gamma}^4 * \tilde{\gamma}^1, \\ \tilde{\gamma}^2 * \tilde{\gamma}^4 + \tilde{\gamma}^4 * \tilde{\gamma}^2 &= -r^{-1}, \quad r^{-1} \tilde{\gamma}^3 * \tilde{\gamma}^4 = -r \tilde{\gamma}^4 * \tilde{\gamma}^3, \quad r \tilde{\gamma}^3 * \tilde{\gamma}^2 = -r^{-1} \tilde{\gamma}^2 * \tilde{\gamma}^3.\end{aligned}$$

Provided permutation rules for the generators and their action on the basic elements $e^{(\alpha)}$ are known, it is possible to evaluate the result of the action of an arbitrary differential operator from $\widetilde{\text{Diff}}(\mathcal{W})$ on an arbitrary element of $\widetilde{\mathcal{W}}$. It is easy to see that $\tilde{\partial}_\mu * e^{(\alpha)} = 0$. Further we find $\tilde{\gamma}^\mu * e^{(\alpha)} = (\Phi_1 \gamma^\mu) \cdot (\Phi_2 e^{(\alpha)})$. In general, the correspondence

$$\tilde{\pi}(\tilde{A}) = \pi(\Phi_1 A) \rho(\Phi_2) = \rho(\Phi_1^{(1)}) \pi(A) \rho(S(\Phi_1^{(2)}) \Phi_2),$$

where we explicitly exposed the representations π and ρ of the algebras \mathcal{A} and \mathcal{H} on \mathbf{W} , defines a representation of the twisted module-algebra $\widetilde{\text{Lin}}(\mathbf{W}, \mathbf{W})$ in the ordinary matrix ring $\text{Lin}(\mathbf{W}, \mathbf{W})$, and that may serve as another confirmation to the thesis about equivalence between twisted Hopf algebras as internal symmetries. Note that the isomorphism mentioned is a homomorphism of the $\tilde{\mathcal{H}}$ -modules. For the Dirac matrices we have

$$\tilde{r}(\tilde{\gamma}^1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ r & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad \tilde{r}(\tilde{\gamma}^2) = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & r^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix},$$

$$\tilde{r}(\tilde{\gamma}^3) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & r & 0 & 0 \end{pmatrix}, \quad \tilde{r}(\tilde{\gamma}^4) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ r^{-1} & 0 & 0 & 0 \end{pmatrix},$$

These matrices obey the same commutation relations as $\tilde{\gamma}$.

For calculating scattering matrix elements it is necessary to know how to evaluate traces of twisted matrices. Trace is a Lorentz-invariant linear functional on $\text{Lin}(\mathbf{W}, \mathbf{W})$ which realizes a homomorphism $\text{t}: \mathcal{M} \otimes \text{Lin}(\mathbf{W}, \mathbf{W}) \rightarrow \mathcal{M}$ between two \mathcal{H} -covariant \mathcal{M} -modules, and that is true for the twisted case:

$$\tilde{\text{t}}(\tilde{a} * \tilde{A}) = \text{t}(\Phi_1 a \cdot \Phi_2 A) = (\Phi_1 a) \text{t}(\Phi_2 A) = (\Phi_1 a) \varepsilon(\Phi_2) \text{t}(A) = \tilde{a} * \tilde{\text{t}}(\tilde{A})$$

for $a \in \mathcal{M}$ and $A \in \text{Lin}(\mathbf{W}, \mathbf{W})$. The functional $\tilde{\text{t}}$ can be evaluated through representation $\tilde{\pi}$ by the formula

$$\tilde{\text{t}}(\tilde{A}) = \text{t}(A) = \text{Tr}[r(A)] = \text{Tr}[\tilde{\pi}(\Phi_1^{-1} A) \rho(\Phi_2^{-1})].$$

This expression can be reduced to a more closed expression held for the $U_{1,r}(so(1,3))$ -symmetry: $\tilde{\text{t}}(\tilde{A}) = \text{Tr}[\tilde{\pi}(\tilde{A}) \rho(\omega)]$, where $\omega \in \mathcal{H}$ is equal to $\Phi_1^{-1} S(\Phi_2^{-1}) S(\Phi_{2'}) \Phi_{1'}$. Involved in the expression for ω are the elements taking part in definition of the twisted antipode \tilde{S} , hence $\omega = 1$ and $\tilde{\text{t}}(\tilde{A}) = \text{Tr}[\tilde{\pi}(\tilde{A})]$. Similar statement is true whenever the twist obeys the factorization property $(\Delta \otimes id)(\Phi) = \Phi_{23} \Phi_{13}$, as that is the case with the twisted tensor product.

Calculating the trace of the product of two invariant matrices $(\tilde{p}, \tilde{\gamma})_r$ and $(\tilde{q}, \tilde{\gamma})_r$, where entities \tilde{q}_μ are permuted by the same rules as \tilde{p}_μ , gives the invariant scalar product $(\tilde{p}, \tilde{q})_r$, as might be expected, having in mind the links between objects of the twist-related geometries

Now let us turn to the Dirac equation

$$i\gamma^\mu \partial_\mu \psi(z) = m\psi(z). \tag{11}$$

Differential operator $D = i\gamma^\mu \cdot \partial_\mu$ is invariant under the classical Lorentz algebra action, therefore the quantum Dirac operator is

$$\tilde{D} = r^{-1}\tilde{\gamma}^1 * \tilde{\partial}_1 + r\tilde{\gamma}^2 * \tilde{\partial}_2 + r^{-1}\tilde{\gamma}^3 * \tilde{\partial}_3 + r\tilde{\gamma}^4 * \tilde{\partial}_4.$$

It is preserved by the action of the twisted algebra $U_{1,r}(so(1,3))$. Eigenvalue functions for the classical Dirac operator in the form of plane waves read

$$\begin{aligned}\psi^{(\alpha)}(z) &= e^{-i(p,z)} \cdot u^{(\alpha)}(p), \quad \text{for positive energies,} \\ \psi^{(\alpha)}(z) &= e^{i(p,z)} \cdot v^{(\alpha)}(p), \quad \text{for negative energies,}\end{aligned}$$

where the spinors $u^{(\alpha)}(p)$ and $v^{(\alpha)}(p)$ in the chiral representation are decomposed via the basic elements $e^{(\alpha)}$:

$$\begin{aligned}u^{(1)}(p) &= (p_1 + m) \cdot e^{(1)} + p_4 \cdot e^{(2)} + (p_3 + m) \cdot e^{(3)} - p_4 \cdot e^{(4)}, \\ u^{(2)}(p) &= p_2 \cdot e^{(1)} + (p_3 + m) \cdot e^{(2)} - p_2 \cdot e^{(3)} + (p_1 + m) \cdot e^{(4)}, \\ v^{(1)}(p) &= (p_1 + m) \cdot e^{(1)} + p_4 \cdot e^{(2)} - (p_3 + m) \cdot e^{(3)} + p_4 \cdot e^{(4)}, \\ v^{(2)}(p) &= p_2 \cdot e^{(1)} + (p_3 + m) \cdot e^{(2)} + p_2 \cdot e^{(3)} - (p_1 + m) \cdot e^{(4)}.\end{aligned}$$

To simplify the resulting expressions we use non-normalized spinors $u^{(\alpha)}(p)$, $v^{(\alpha)}(p)$. Because of the Lorentz-invariance of the Dirac operator, the same functions expressed through the generators in terms of the new multiplication will be solutions to the deformed Dirac equation. The invariant factors $e^{\pm i(p,z)}$ are separated as multipliers $e^{\pm i(\tilde{p},\tilde{z})r}$. Ultimately, we obtain the full set of the eigenvalue functions of the quantum Dirac operator:

$$\begin{aligned}\tilde{\psi}^{(\alpha)}(\tilde{z}) &= e^{-i(\tilde{p},\tilde{z})r} * \tilde{u}^{(\alpha)}(\tilde{p}), \quad \text{for positive energies,} \\ \tilde{\psi}^{(\alpha)}(\tilde{z}) &= e^{i(\tilde{p},\tilde{z})r} * \tilde{v}^{(\alpha)}(\tilde{p}), \quad \text{for negative energies}\end{aligned}$$

with

$$\tilde{u}^{(1)}(\tilde{p}) = (r\tilde{p}_1 + m) * e^{(1)} + r\tilde{p}_4 * e^{(2)} + (\tilde{p}_3 + m) * e^{(3)} - \tilde{p}_4 * e^{(4)},$$

$$\begin{aligned}
\tilde{u}^{(2)}(\tilde{p}) &= r\tilde{p}_2 * e^{(1)} + (r\tilde{p}_3 + m) * e^{(2)} - \tilde{p}_2 * e^{(3)} + (\tilde{p}_1 + m) * e^{(4)}, \\
\tilde{v}^{(1)}(\tilde{p}) &= (r\tilde{p}_1 + m) * e^{(1)} + r\tilde{p}_4 * e^{(2)} - (\tilde{p}_3 + m) * e^{(3)} + \tilde{p}_4 * e^{(4)}, \\
\tilde{v}^{(2)}(\tilde{p}) &= r\tilde{p}_2 * e^{(1)} + (r\tilde{p}_3 + m) * e^{(2)} + \tilde{p}_2 * e^{(3)} - (\tilde{p}_1 + m) * e^{(4)}.
\end{aligned}$$

General solution to (11) can be written down as the sum of the plane waves similarly as for the deformed Klein-Gordon-Fock equation. It can be derived from the classical solution in two different ways. So, the transition to the non-commutative variables may be performed prior to Fourier integration which is then fulfilled via the quantum invariant measure on the mass shell. Another possible way to obtain a solution to the deformed Dirac equation from the classical one is to substitute non-commutative variables into the final classical expression making use of the connection between the twisted and non-twisted monomials and bearing in mind that the Lorentz-invariant factors remain so (as well as the result of multiplication by them) both in the twisted and non-twisted cases. This explains how to deal with the objects of physical interest in proceeding to a twist-related geometry, no matter classical or already quantum the original one is.

5 Conclusion

In conclusion, let us briefly formulate the results of our consideration. We have shown the effectiveness of knowing the structure of the quantum Lorentz algebra for constructing the quantum space-time. Group approach appears very natural from the geometrical point of view, but many nontrivial deformations are described in terms of universal enveloping algebras. In other words, any object of classical geometry transforming in somehow under the symmetry algebra action does exist in the quantum space, too, acquiring a new algebraic content. The problem of description of the algebraic "zoo" of the phase space thus boils down to studying unitary representation of

the function algebra on cotangent bundles.

Twists supply with a powerful tool for investigation of the entire classes of geometries uniformly, and from that point of view it seems natural to start from the simplest representatives. In the case of the Lorentz algebra they are $U(so(1,3))$ and $U_{q;1}(so(1,3))$. Either solutions correspond to mutually commuting right and left $\frac{1}{2}$ -spinors. Performing twist transformations in accordance with the classification scheme of Lorentz algebra quantization, one can reach every other modifications of relativistic geometry, using the advantages of the presented approach. Rich twist-structure of deformations of the Lorentz algebra itself makes it possible, in prospective, to use this technique for building curved spaces of general relativity, where no translations but only the Lorentz boosts and rotations survive.

Acknowledgements. One of the authors (PPK) thanks Laurent Baulieu for the kind invitation in the framework of the jumelage programme, and the hospitality in LPTHE of Université Pierre et Marie Curie.

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